EXTENDED-DOMAIN METHOD IN HEAT-CONDUCTION PROBLEMS

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An effective method for the solution of mathematical physics problems is discussed in the example of heat-conduction problems. Numerical computations are carried out to illustrate the accuracy and convergence of the method.

The basic idea of the method of extended domains is utilized primarily in elasticity problems [1-3], although even in that area it has not been developed and applied to the extent that it should.

The method is based on a simple physical notion; it is extremely general and is well suited to computer implementation. It can be used effectively for the solution of a broad sphere of problems in mathematical physics. The substance of the method is as follows. A given domain D is immersed (conditionally) in a certain extended domain D', for which the fundamental solution of the initial differential equation of the problem is known. The surface Γ bounding D is treated in D' as a function of distributed sources, the strength of which is to be determined. If that surface is partitioned into n finite elements and it is assumed that the density of sources inside each element is, say, uniformly distributed there, then by satisfying the boundary conditions at the midpoints of the surface elements we obtain a system of nondegenerate algebraic equations in the unknown densities.

In the investigation of nonsteady problems it is necessary to introduce additional time quantization, adopting a priori a certain distribution function for the densities of sources in application to each discrete time interval.

The extended-domain method has two important advantages: 1) The system of equations is formed solely in terms of points of the boundary surface; 2) for each type of domain (plane, axisymmetric, three-dimensional) the algorithm is general, regardless of the configuration of the domain or the type of boundary conditions.

Below, without sacrificing generality, we consider the planar heat-conduction problem for an arbitrary anisotropic domain D bounded by a contour Γ. Let the required function v(x, y, t) satisfy the following equation and initial and boundary conditions:

$$\frac{\partial v}{\partial t} = a_1 \frac{\partial^2 v}{\partial x^2} + a_2 \frac{\partial^2 v}{\partial y^2} \text{ in } D \text{ for } t > 0, \tag{1}$$

$$v = \varphi(x, y) \text{ in } D \text{ for } t = 0, \tag{2}$$

$$v - R(x, y)\left[a_1 \frac{\partial v}{\partial x}\cos(h, x) + a_2 \frac{\partial v}{\partial y}\cos(h, y)\right] =$$

$$= f(x, y, t) \text{ on } \Gamma \text{ for } t > 0. \tag{3}$$

It is assumed here that the coordinate axes coincide with the directions of the principal anisotropy axes, a_1 and a_2 are constants, E, φ, and f are given functions, and h is the normal to the contour Γ.

We take as the extended domain D' the unbounded plane, for which the known fundamental solution

$$G(x, y, t) = \frac{1}{4\pi (a_1 a_2)^{1/2}(t-\tau)} \exp\left[-\frac{(x-x')^2}{4a_1(t-\tau)} - \frac{(y-y')^2}{4a_2(t-\tau)}\right]$$

is interpreted as the temperature induced in the unbounded plane by the instantaneous release of heat by a point source of unit strength at a point (x', y') of the plane D' at time t = τ.*

*In a number of situations it may be practical to use other extended domains D' (for example, a halfplane, strip, etc.) and fundamental solutions in the form of Green (for R = 0), Neumann (R → ∞), or Robin (0 < R < ∞) functions.

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We partition the contour Γ into n sufficiently small rectilinear segments and assume that q_{jp} is the strength of sources distributed uniformly on each j-th segment and acting continuously for a time interval $[t_{p-1}, t_p]$ ($j = 1, n$; $p = 1, k$). Then the temperature of an arbitrary point of the domain $D \subset D'$ at time $t = t_k$, satisfying Eq. (1) and the initial condition (2), is determined from the expression

$$v(x, y, t_k) = v_0(x, y, t_k) + \sum_{p=1}^{k}\sum_{j=1}^{n} q_{jp} I_{jp}(x, y, t_k), \tag{4}$$

in which

$$I_{jp} = \frac{1}{4\pi (a_1 a_2)^{1/2}} \int_{t_{p-1}}^{t_p} \frac{d\tau}{t-\tau} \int_0^{l_j} \exp\left[-\frac{a_2(x-x')^2 + a_1(y-y')^2}{4a_1a_2(t-\tau)}\right] d\varepsilon;$$

$$v_0 = \iint_D \varphi(\xi, \eta)\, G(x-\xi, y-\eta, t_k)\, d\xi d\eta;$$

$$l_j = [(x_j - x_{j-1})^2 + (y_j - y_{j-1})^2]^{1/2};$$

$$x' = x_{j-1} + \alpha_j\varepsilon;\ y' = y_{j-1} + \beta_j\varepsilon;$$

$$\alpha_j = (x_j - x_{j-1})/l_j;\ \beta_j = (y_j - y_{j-1})/l_j.$$

It is convenient from the standpoint of diminishing the number of equations and increasing the accuracy to choose the segment lengths l_j depending on the smoothness of the contour and the boundary conditions, namely shorter lengths where the curvature is greater or where the boundary functions are joined or suffer discontinuities.

After integration the function I_{jp} and its derivatives with respect to the coordinates assume the form

$$I_{jp} = \frac{1}{4\pi m_j \delta a_1} \sum_{s=1}^{2} (-1)^s \left[\eta_j F_{jp}^{(s)} + \xi_{js} \sum_{r=1}^{2} (-1)^r \Psi_{jp}^{(r,s)}\right], \tag{5}$$

$$\frac{\partial I_{jp}}{\partial x} = -\frac{1}{4\pi m_j a_1} \sum_{s=1}^{2} (-1)^s \left[\beta_j F_{jp}^{(s)} + \delta\alpha_j \sum_{r=1}^{2} (-1)^r \mathrm{Ei}(-\nu_{rs}^2)\right], \tag{6}$$

$$\frac{\partial I_{jp}}{\partial y} = -\frac{1}{4\pi m_j a_1} \sum_{s=1}^{2} (-1)^s \left[-\alpha_j F_{jp}^{(s)} + \delta\beta_j \sum_{r=1}^{2} (-1)^r \mathrm{Ei}(-\nu_{rs}^2)\right],$$

$$m_j = \beta_j^2 + \delta^2\alpha_j^2;\ \delta^2 = a_2/a_1$$

is the anisotropy parameter;

$$\eta_j = \delta[\alpha_j(y - y_{j-1}) - \beta_j(x - x_{j-1})];$$

$$\xi_{j1} = \beta_j(y - y_{j-1}) + \delta^2\alpha_j(x - x_{j-1});\ \xi_{j2} = \xi_{j1} - m_j l_j;$$

$$F_{jp}^{(s)} = 2\sqrt{\pi}\, \lambda_{js} \int_{\mu_{1s}}^{\mu_{2s}} \Phi(\mu) \exp(-\lambda_{js}^2\mu^2)\, d\mu;$$

$$\lambda_{js} = \eta_j/\xi_{js};$$

$$\mu_{1s} = \frac{\xi_{js}}{2\delta[m_j a_1(t_k - t_{p-1})]^{1/2}};\ \mu_{2s} = \mu_{1s}\left(\frac{t_k - t_{p-1}}{t_k - t_p}\right)^{1/2};$$

$\Phi(\mu) = 2/\sqrt{\pi}\int_0^{\mu} \exp(-u^2)du$ is the probability integral;

$$\Psi_{jp}^{(r,s)} = \frac{\sqrt{\pi}\,\Phi(\mu_{rs})}{\mu_{rs}} \exp(-\lambda_{js}^2\mu_{rs}^2) - \mathrm{Ei}(-\nu_{rs}^2);$$

$$\mathrm{Ei}(-\nu_{rs}^2) = \int_{\nu_{rs}^2}^{\infty} \frac{\exp(-u)}{u}\, du$$

is the integral exponential function; and

$$\nu_{rs} = |\mu_{rs}| (1 + \lambda_{js}^2)^{1/2}.$$

To simplify the calculations it is useful to approximate the probability integral, for example in the form

$$\Phi(\mu) = \begin{cases} \frac{\mu}{\sqrt{\pi}} [\exp(-c_1\mu^2) + \exp(-c_2\mu_2^2)] & \text{for } |\mu| < 2.2, \\ \operatorname{sgn}\mu & \text{for } |\mu| \geqslant 2.2, \end{cases}$$

where $c_1 = 0.060025$ and $c_2 = 0.588787645$. The error of approximation in this case is not greater than 0.25%.

As a result, the function $F_{jp}^{(s)}$ can be calculated according to the equation

$$F_{jp}^{(s)} = \sum (-1)^r [\pi \operatorname{sgn} A_{rs} \Phi(\lambda_{js} A_{rs}) - \lambda_{js} \operatorname{sgn} B_{rs}^2 Q(\lambda_{js}, B_{rs})],$$

$$Q(\lambda_{js}, B_{rs}) = \frac{1}{c_1 + \lambda_{js}^2} \exp[-B_{rs}^2 (c_1 + \lambda_{js}^2)] + \frac{1}{c_2 + \lambda_{js}^2} \exp[-B_{rs}^2 (c_2 + \lambda_{js}^2)];$$

$$A_{rs} = \mu_{rs} \text{ and } B_{rs} = 0 \text{ for } |\mu_{rs}| \geqslant 2.2;$$

$$A_{1s} = 2.2 \operatorname{sgn} \mu_{1s},\ A_{2s} = \mu_{2s},\ B_{1s} = \mu_{1s} \text{ and}$$
$$B_{2s} = 2.2 \operatorname{sgn} \mu_{2s} \text{ for } |\mu_{1s}| < 2{,}2 < |\mu_{2s}|;$$
$$A_{rs} = 0 \text{ and } B_{rs} = \mu_{rs} \text{ for } |\mu_{rs}| \leqslant 2{,}2 \ (s,\ r = 1{,}2).$$

If we require that the boundary conditions (3) be satisfied successively at times $t = t_m$ ($m = \overline{1, k}$) at the midpoints of the contour Γ:

$$x_{0i} = \frac{1}{2}(x_{i-1} + x_i),\quad y_{0i} = \frac{1}{2}(y_{i-1} + y_i),$$

we obtain the recursive system of equations

$$\sum_{j=1}^{n} c_{ijm}^{(m)} q_{jm} = D_{im} - \sum_{p=1}^{m-1} \sum_{j=1}^{n} c_{ijp}^{(m)} q_{jp} \ (i = \overline{1, n})$$

or, in matrix form,

$$[c_m^{(m)}] \{q_m\} = \{D_m\} - \sum_{p=1}^{m-1} [c_p^{(m)}] \{q_p\},$$

where

$$c_{ijp}^{(m)} = I_{jp}(x_{0i}, y_{0i}, t_m) \pm Ra_1 \left(\beta_i \frac{\partial I_{jp}}{\partial x} - \delta^2 \alpha_i \frac{\partial I_{jp}}{\partial y} \right);$$

$$D_{im} = f(x_{0i}, y_{0i}, t_m) - v_0(x_{0i}, y_{0i}, t_m) \mp Ra_1 \left(\beta_i \frac{\partial v_0}{\partial x} - \delta^2 \alpha_i \frac{\partial v_0}{\partial y} \right).$$

The upper signs correspond to the outer contour of the domain D, and the lower signs to the inner contour, in the clockwise direction. The diagonal elements of the principal matrix, being in fact the maximum in the row, are computed according to the expression

$$c_{ijm}^{(m)} = \frac{l_i}{4\pi\delta a_1} \left[\frac{\sqrt{\pi}\Phi(\mu_i^*)}{\mu_i^*} - \operatorname{Ei}(-\mu_i^{*2}) \right] + \frac{R}{2},$$

where

$$\mu_i^* = \frac{m_i l_i}{4\delta [m_i a_1 (t_m - t_{m-1})]^{1/2}}.$$

The number of equations of the system can be reduced if the problem has a plane of symmetry. When a uniform time step is used up to the time t_k, it is sufficient to compute the matrices $[c_1^{(m)}]$ ($m = \overline{1, k}$), since $[c_1^{(1)}] = [c_2^{(2)}] = \ldots = [c_k^{(k)}]$, $[c_1^{(2)}] = [c_2^{(3)}] = \ldots = [c_{k-1}^{(k)}]$, etc. Then the solution can be represented in the form

$$\{q_m\} = [c_1^{(1)}]^{-1} \left\{ \{D_m\} - \sum_{p=1}^{m-1} [c_1^{(m-p+1)}] \{q_p\} \right\}.$$

In the limiting steady-state situation where the source function is

$$G(x, y) = -\frac{1}{2\pi\delta a_1} \ln[\delta^2(x-x')^2 + (y-y')^2],$$

the required solution is represented by the sum

$$v(x, y) = \sum_{j=1}^{n} q_j I_j(x, y),$$

where

$$\{q\} = [c]^{-1}\{D\};$$

$$I_j(x, y) = \frac{1}{2\pi m_j \delta a_1} \sum_{s=1}^{2} (-1)^s \xi_{js} \left(\ln \rho_{js} + \lambda_{js} \operatorname{arctg} \frac{1}{\lambda_{js}} - 1\right);$$

$$\rho_{js} = |\xi_{js}|(1 + \lambda_{js}^2)^{1/2}; \; D_i = f(x_{0i}, y_{0i});$$

$$c_{ij} = I_j(x_{0i}, y_{0i}) \pm Ra_1 \left[\beta_i \frac{\partial I_j(x_{0i}, y_{0i})}{\partial x} - \delta^2 \alpha_i \frac{\partial I_j(x_{0i}, y_{0i})}{\partial y}\right];$$

$$\frac{\partial I_j}{\partial x} = \frac{1}{2\pi m_j a_1} \sum_{s=1}^{2} (-1)^s \left(\delta \alpha_j \ln \rho_{js} - \beta_j \operatorname{arctg} \frac{1}{\lambda_{js}}\right);$$

$$\frac{\partial I_j}{\partial y} = \frac{1}{2\pi m_j a_1} \sum_{s=1}^{2} (-1)^s \left(\delta^{-1} \beta_j \ln \rho_{js} + \alpha_j \operatorname{arctg} \frac{1}{\lambda_{js}}\right);$$

and the quantities ξ_{js} and λ_{js} are the same as in expressions (5) and (6). The diagonal elements of the matrix [c] are computed according to the equation

$$c_{ii} = \frac{l_i}{2\pi a_1 \delta}\left(1 - \ln \frac{m_i l_i}{2}\right) + \frac{R}{2} \; (i = \overline{1, n}).$$

To illustrate the convergence and accuracy of the given algorithms we have selected to examples for implementation on an M-222 computer.

Table 1 summarizes the results of computing the function v(x, y) for the well-known steady-state problem in which the domain D is bounded by the equilateral triangle with unit side and at the boundary the functions R(x, y) = 0 and $f = x^2 + y^2$ are given. This type of problem occurs in a number of applications of mathematical physics.

The third and fourth columns of the table give the values of the function for partitions of the sides of the triangle into, respectively, 10 and 20 equal segments of length $l_j = 0.1$ and $l_j = 0.05$, and the fifth column corresponds to the values of the functions for a nonuniform step: $l_j = 0.025$ over one fourth the length from the vertex of the triangle and $l_j = 0.075$ over the remaining length, with preservation of the number of segments n = 20. For comparison the last column gives the exact values of the function. The origin of the coordinate system Oxy was placed at the midpoint of the base of the triangle, with the axis Ox running along the base and the axis Oy along the altitude.

In the implementation of nonsteady problems with the required accuracy, the degree of time quantization depends mainly on the nature of the time variation of the boundary conditions. This fact is illustrated below in a test problem.

Table 2 lists the errors (%) in the calculation of the temperature at points of an isotropic ($a_1 = a_2 = a$) half-plane $x \geq 0$, on the surface of which the temperature is given by one of the functions: f_1 = const; $f_2 = t$; $f_3 = t^2$. The temperature is estimated for four values of the Fourier parameter ($Fo = at/x^2$) and two time-quantization schemes.

It is important to note that the computing time can be shortened while simultaneously attaining a higher accuracy by applying a certain prediction procedure (formula) to the results of a few successive solutions with an increasing number of steps.

TABLE 1. Comparison of Computed Values of Steady-State Function $v(x, y)$ with Exact Values for Three Quantizations of the Boundary of a Triangular Domain

x	y	$n=10$	$n=20$		Exact solution
		$l_j=0,1$	$l_j=0,05$	variable step	
0	0,05	0,041093	0,040962	0,040950	0,040945
0	0,1	0,077919	0,077776	0,077759	0,077757
0	$\sqrt{3}/6$	0,194634	0,194467	0,194446	0,194444
0	0,6	0,409435	0,409076	0,409020	0,409030
0	0,8	0,645901	0,645376	0,644156	0,644026
0,3	$\sqrt{3}/6$	0,194558	0,194464	0,194461	0,194444

TABLE 2. Errors (%) for Three Laws of Heating of a Halfplane and Two Time-Quantization Schemes

k	f			
	const	t	t^2	F_0
10	—2,20	1,77	3,54	1,1
20	—1,00	0,92	1,75	
10	—1,58	1,03	2,84	2,2
20	—0,72	0,49	1,12	
10	—0,78	0,57	1,91	10
20	—0,35	0,22	0,75	
10	—0,57	0,45	1,15	20
20	—0,25	0,18	0,64	

NOTATION

D	is the given domain;
Γ	is the boundary of given domain;
D'	is the extended domain;
v	is the unknown function;
x, y, x', y'	are the coordinates on the plane;
t, τ,	is the time;
φ	is the initial condition;
f	is the boundary condition;
R	is the thermal resistance;
h	is the normal to domain boundary;
a_1, a_2	are the thermal diffusivities along principal axes of anisotropy;
l	is the length of contour segment;
q	is the source strength;
n	is the number of contour segments;
k	is the number of time steps;
x_j, y_j, x_i, y_i	are the coordinates of contour points.

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